2024 AP Calculus AB Free Response Questions
Section II, Part A (30 minutes)
# of questions: 2

A graphing calculator may be used for this part.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

1. The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table above.
   (a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \leq t \leq 7$. Show the work that leads to your answer and include units of measure.
   (b) Use a left Riemann sum with the three subintervals indicated by the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.
   (c) For $12 \leq t \leq 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \frac{24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the set up for your calculations.
   (d) For the model defined in part (c), it can be shown that $C''(t) = \frac{24.55e^{0.01}\ (100-t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

(a) $C'(5) = \frac{C(7)-C(3)}{7-3} = \frac{69-85}{4} = -\frac{16}{4} = -4\ °\text{C/min}$

(b) $\int_0^{12} C(t)\,dt = 3(100) + 4(85) + 5(69) = 985$
This is the average value of $C(t)$ on the interval $0 \leq t \leq 12$ –
the average temperature of the coffee, in °C, for the first 12 minutes.

(c) $C(20) = 55 + \int_{12}^{20} \frac{-24.55e^{0.01t}}{t}\,dt = 40.329°\text{C}.$

(d) For $12 < t < 20$, $C''(t) < 0$: the numerator will be positive (the exponential is positive, $100-t$ is positive, and 24.55 – their product will be positive). The denominator $t^2 > 0$ for all $t \neq 0$. Therefore, $C''(t)$ will be positive, therefore C' is increasing.

2. A particle moves along the x-axis so that its velocity at time $t \geq 0$ is given by $v(t) = \ln(t^2 - 4t + 5) - 0.2t$.

(a) There is one time, $t = t_R$, in the interval $0 < t < 2$ when the particle is at rest (not moving). Find t_R. For $0 < t < t_R$, is the particle moving to the right or to the left? Give a reason for your answer.

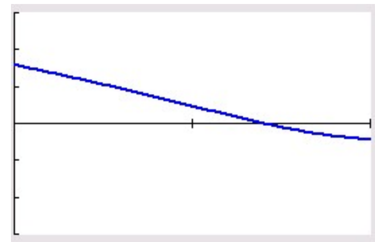
(b) Find the acceleration of the particle at time $t = 1.5$. Show the setup for your calculations. Is the speed of the particle increasing or decreasing at time $t = 1.5$? Explain your reasoning.

(c) The position of the particle at time t is $x(t)$, and its position at time $t = 1$ is $x(1) = -3$. Find the position of the particle at time $t = 4$. Show the setup for your calculations.

(d) Find the total distance traveled by the particle over the interval $1 \leq t \leq 4$. Show the setup for your calculations.

(a) At rest when $v(t) = 0$: $\ln(t^2 - 4t + 5) - 0.2t = 0$
Graphing the function (window [0,2,1], [−3, 3, 1]), we get the graph to the right.
Tracing the x−intercept, we get $x = 1.426$.

Since $v(t)$ is above the x−axis ($v(t) > 0$, the particle is moving to the right.



(b) $a(t) = v'(t) \rightarrow \ a(1.5) = \frac{d}{dt}[\ln(t^2 - 4y + 5) - 0.2t]_{t=1.5} = -0.99999 = -1$
Since $v(1.5) < 0$ and $a(1.5) < 0$, then the speed is increasing.

(c) $x(4) = x(1) + \int_1^4 v(t)\,dt = -2.803$ units

(d) $X_{tot}(t) = \int_1^4 |v(t)|\,dt = 0.958$ units

Section II, Part B (1 hour)

# of questions: 4

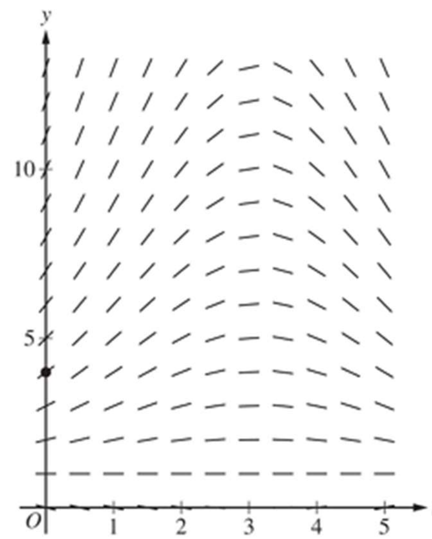
A graphing calculator may NOT be used for this part.

3. The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon ($t = 0$). It is known that $H(0) = 4$.

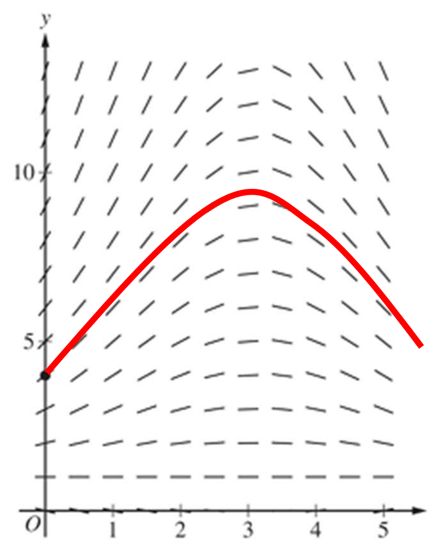
   (a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point (0, 4).

   (b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

   (c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation
   $$\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$$
   with initial condition $H(0) = 4$.



(a) See graph.



(b) $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right) = 0$

$H = 1 \quad \cos\frac{t}{2} = 0 \rightarrow \frac{t}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \quad \rightarrow \quad t = \pi, ~~3\pi~~, ~~5\pi~~, \ldots$

The only solution is at $t = \pi$.

For $0 < t < \pi$: $\cos\frac{t}{2} > 0 \quad \therefore$ *increasing*

For $t > \pi$: $\cos\frac{t}{2} < 0 \quad \therefore$ *decreasing* $\quad \rightarrow$ *it is a rel max*

(c) $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$

$$\frac{dH}{H-1} = \frac{1}{2}\cos\left(\frac{t}{2}\right)\, dt$$

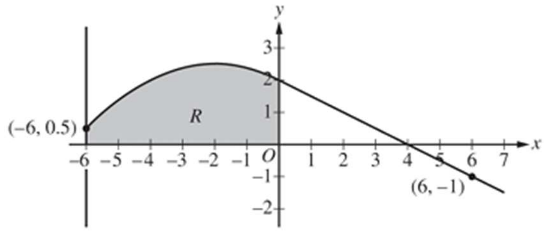
$$\int \frac{dH}{H-1} = \frac{1}{2}\int \cos\left(\frac{t}{2}\right)\, dt$$

$$\ln|H-1| = \sin\left(\frac{t}{2}\right) + C$$

at $t = 0$: $\ln|4-1| = \sin 0 + C \rightarrow C = \ln 3$

$$\ln|H-1| = \sin\left(\frac{t}{2}\right) + \ln 3 \quad \rightarrow \quad H - 1 = \pm e^{\sin\left(\frac{t}{2}\right) + \ln 3} \quad \rightarrow H = 1 \pm 3e^{\sin\left(\frac{t}{2}\right)}$$

Since $H(0) = 4$: $H(t) = 1 + 3e^{\sin\left(\frac{t}{2}\right)}$



Graph of f

4. The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the $x-$ and $y-$axes. Region R has an area of 12.

(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.\

(b) For the function g defined in part (a), find all values of x in the interval $0 \le x \le 6$ at which the graph of g has a critical point. Give the reason for your answer.

(c) The function h is defined $h(x) = \int_{-6}^x f'(t)\,dt$ Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answer.

(a) $g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^0 f(t)\,dt = -12$

$g(4) = \displaystyle\int_0^4 f(t)\,dt = \frac{1}{2}(2)(4) = 4$ (use the area of a triangle)

$g(6) = \displaystyle\int_0^4 f(t)\,dt + \int_4^6 f(t)\,dt = 4 - \frac{1}{2}(1)(2) = 3$

(b) The graph of f is also the graph of $g'(x)$, therefore, where it crosses the $x-$axis is a critical point. This occurs at $x = 4$. Since it crosses from positive to negative, $x = 4$ is a relative maximum.

(c) $h(6) = \int_{-6}^6 f'(t)\,dt = 12 + 3 = 15$

$h'(6) = \dfrac{d}{dx}\displaystyle\int_{-6}^6 f'(t)\,dt = \text{slope of line} \rightarrow \dfrac{-1-2}{6} = -\dfrac{3}{6} = -\dfrac{1}{2}$

$h''(6) = f''(6) = 0 \rightarrow h'(x)$ is a constant from $0 < x < 7$, therefore $h''(6) = 0$

5. Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\frac{dy}{dx} = \frac{-2x}{3+4y}$.

(a) There is a point on the curve near (2, 4) with x–coordinate 3. Use the line tangent to the curve at (2, 4) to approximate the y–coordinate of this point.

(b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

(c) The curve intersects the positive x–axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

(d) For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is a point (4,2), the y–coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x–coordinate of the particle's position with respect to time?

(a) $\frac{dy}{dx} = -\frac{2x}{3+4y} \rightarrow$ At (2,4): $\frac{dy}{dx} = -\frac{4}{3+16} = -\frac{4}{19} \rightarrow \quad y - 4 = -\frac{4}{19}(x-2) \rightarrow \quad y = -\frac{4}{19}x + \frac{84}{19}$

at $x = 3$: $y = -\frac{12}{19} + \frac{84}{19} = \frac{72}{19}$

(b) At $y = 1$: $x^2 + 3 + 2 = 48 \rightarrow x^2 = 43 \rightarrow x = \pm\sqrt{43}$

$$\frac{dy}{dx} = -\frac{2x}{7} = 0 \rightarrow x = 0 \qquad 0^2 + 3(1) + 2(1)^2 = 48? \quad NO$$

(c) $At\ (\sqrt{48}, 0)$: $\frac{dy}{dx} = -\frac{2(\sqrt{48})}{3+4(0)} = -\frac{2\sqrt{48}}{3} \rightarrow Not\ vertical$

(d) $3y^2\frac{dy}{dt} + 2\left[x\frac{dy}{dt} + y\frac{dx}{dt}\right] = 0$

$$3y^2\frac{dy}{dt} + 2x\frac{dy}{dx} + 2y\frac{dx}{dt} = 0$$

$$At\ (4,2), \frac{dy}{dt} = -2:\ 3(2)^2(-2) + 2(4)(-2) + 2(2)\frac{dx}{dt} = 0$$

$$-24 - 16 + 4\frac{dx}{dt} = 0$$

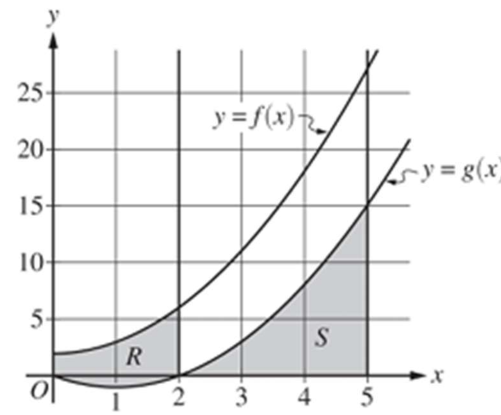
$$4\frac{dx}{dt} = 40 \quad \rightarrow \quad \frac{dx}{dt} = 10$$

6. The functions f and g are defined by $f(x) = x^2 + 2$ and $g(x) = x^2 - 2x$, as shown in the graph.

(a) Let R be the region bounded by the graphs of f and g, from $x = 0$ to $x = 2$, as shown in the graph. Write, but do not evaluate, an integral expression that gives the area of the region R.

(b) Let S be the region bounded by the graph of g and the x–axis, from $x = 2$ to $x = 5$, as shown in the graph. Region S is the base of a solid. For this solid, at each x the cross section perpendicular to the x–axis is a rectangle with height equal to half its base in region S. Find the volume of the solid. Show the work that leads to your answer.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when region S, as described in part (b), is rotated about the horizontal line $y = 20$.



(a) $A = \int_0^2 [f(x) - g(x)]\, dx = \int_0^2 [2x + 2] dx$

(b) Base: $b(x) = g(x) = x^2 - 2x$

Cross Section: Rectangle – A = bh → $A(x) = (x^2 - 2x)(\frac{1}{2}(x^2 - 2x) = \frac{1}{2}(x^2 - 2x)^2$

$V(x) = \frac{1}{2}\int_2^5 (x^4 - 4x^3 + 4x^2)\ dx = \frac{1}{2}\left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right]\Bigg|_2^5 = \frac{1}{2}\left[\left(625 - 625 + \frac{500}{3}\right) - \left(\frac{32}{5} - 16 + \frac{32}{3}\right)\right]$

$V = \frac{1}{2}\left(\frac{500}{3} - \frac{96-240+160}{15}\right) = \frac{1}{2}\left(\frac{2500}{15} - \frac{16}{15}\right) = \frac{1}{2}\left(\frac{2484}{15}\right) = \frac{1242}{15} = \frac{414}{5}$

(c) $V = \pi \int_a^b R^2 - r^2\ dx = \pi \int_2^5 \left[(20^2) - \left(20 - g(x)\right)^2\right] dx$