**Chapter 5 – Analytic Trigonometry**
**5.1 Verifying Trigonometric Identities**

Def: An *identity* is an equation that is always true regardless of the value(s) substituted.

**Fundamental Identities for Trigonometry** (*Please MEMORIZE!*)

Reciprocal Identities:

$\sin x = \frac{1}{\csc}$ $\cos x = \frac{1}{\sec x}$ $\tan x = \frac{1}{\cot x}$

$\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\cos x}$ $\cot x = \frac{1}{\tan x}$

Quotient Identities:

$\tan x = \frac{\sin x}{\cos x}$ $\cot x = \frac{\cos x}{\sin x}$

Pythagorean Identities:

$\sin^2 x + \cos^2 x = 1$ ** The most important identity of them all **

$1 + \tan^2 x = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$

Even/Odd Identities:

$\sin(-x) = -\sin x$ $\cos(-x) = \cos x$ $\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$ $\sec(-x) = \sec x$ $\cot(-x) = -\cot x$

*These Fundamental Identities are used to verify (or prove) any identity.*

**Advice**
Always start on the side more complicated!

Ex. Prove: $\sec x \cot x = \csc x$

Prove: $\sin x \tan x + \cos x = \sec x$

Ex. Prove: $\cos x - \cos x \sin^2 x = \cos^3 x$

Prove: $\frac{1+\sin}{\cos x} = \sec x + \tan x$

Ex. Prove: $\frac{\sin x}{1+\cos} = \frac{1-\cos}{\sin x}$

Prove: $\frac{\cos x}{1+\sin} + \frac{1+\sin x}{\cos x} = 2\sec x$

Prove: $\frac{1}{1+\cos} + \frac{1}{1-\cos x} = 2 + 2\cot^2 x$

**You are allowed to:**

- Combine terms
- Multiply factors
- Use other known identities
- Separate terms
- Factor
- Simplify radicals
- Change to sines and cosines
- You may not "move" any function from one side to the other
- You may only add 0 or multiply/divide by 1
- These will be in "disguise"

**Advice**

- Work on one side of the identity until you can't go any further. Then work on other side.
- DO NOT MOVE ANYTHING FROM ONE SIDE TO THE OTHER.
- DO NOT MULTIPLY/DIVIDE ON BOTH SIDES

**Homework: Pg. 594 #7, 9, 13, 15, 21, 23, 26, 29, 31, 35 ,45, 53**

**5.2 Sum and Difference Formulas**

**<u>Th</u>: Sum and Difference Formulas:** Given two angles, $\alpha$ and $\beta$, then the following formulas are true:

| **Sum Formulas** | **Difference Formulas** |
|---|---|
| $\sin(\alpha+\beta) = \sin\alpha\, cos \quad + \cos\alpha\sin\beta$ | $\sin(\alpha-\beta) = \sin\alpha\, cos \quad - \cos\alpha\sin\beta$ |
| $\cos(\alpha+\beta) = \cos\alpha\, co \quad - \sin\alpha\sin\beta$ | $\cos(\alpha-\beta) = \cos\alpha\, cos\beta + \sin\alpha\sin\beta$ |
| $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ | $\tan(\alpha-\beta) = \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$ |

Ex. Find the exact value of each of the following:

**<u>Hint</u>**: $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$

1. $\sin 15°$
2. $\cos 75°$
3. $\sin\frac{7\pi}{12}$
4. $\cos 80°\cos 20° + \sin 80°\sin 20°$
5. $\tan 105°$

Ex. Prove the identity: $\frac{\cos(x-y)}{\sin x\cos y} = \cot x + \tan y$

Ex. Suppose $x$ is an angle in Quadrant II with $\sin x = \frac{12}{13}$ and $y$ is an angle in Quadrant I with $\cos y = -\frac{3}{5}$, find each of the following

(a) $\sin y$ (b) $\cos x$ (c) $\sin(x+y)$ (d) $\cos(x-y)$

Ex. Prove the identity: $\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$

Ex. Prove the identity: $\frac{\cos(x+y)}{\cos x \cos y} = 1 + \tan x \tan y$

**Homework: Pg. 603 – 604 #1–8, 9–39 (3's), 40, 45, 50–53, 57–63odd**

**5.3 Double Angle Formulas & Half-Angle Formulas**

Ex. Using the formula for the sin $(x+y)$, write an expression equivalent to $\sin 2x$

Ex. Using the formula for the tan $(x+y)$, write an expression equivalent to $\tan 2x$

Ex. Using the formula for the cos $(x+y)$, write an expression equivalent to $\cos 2x$.

Ex. Write 2 other expressions equivalent to $\cos 2x$.

Ex. If $\sin\theta = \frac{5}{13}$, and θ lies in quadrant II, find the exact value of each of the following:

(a) $\sin 2\theta$ (b) $\cos 2\theta$ (c) $\tan 2\theta$

Ex. Find the exact value of (a) $\frac{2\tan 15°}{1-\tan^2 15°}$ (b) $\cos^2 15 - \sin^2 15$ (c) $\cos 112.5°$

Ex. Prove the identity: $\cos 3x = 4\cos^3 x - 3\cos x$

**Th: Power-Reducing Formulas**

$\sin^2 x = \frac{1-\cos}{2}$ $\cos^2 x = \frac{1+\cos 2x}{2}$ $\sin^2 x = \frac{1-\cos}{1+\cos}$

*These formulas lead us to the following:*

Def: **Half-Angle Formulas**:

1. $\sin\frac{1}{2}x = \pm\sqrt{\frac{1-\cos}{2}}$
2. $\cos\frac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}$
3. $\tan\frac{1}{2}x = \pm\sqrt{\frac{1-\cos}{1+\cos}}$

| |
|---|
| The $\pm$ in the formula indicates you must determine the sign of the expression. The sign is based on the quadrant that the<br><br>**HALF-ANGLE LIES IN.** |

Ex. Find the exact value of $\cos 112.5°$

ex. What is the exact value of cos 105°

Ex. Prove the identity: $\tan x = \frac{1-\cos}{\sin 2}$

**Homework: Pg. 614 – 615 #4 – 68(4's), 69,70**

| |
|---|
| **Day 1: Pg. 614-615#1-15, 17-33odd** |
| **Day 2: Pg. 615 #16-34e, 39-69odd** |
| **Day 3: Pg. 616-617 #70, 72, 77, 79, 80, 81** |

**5.4 Product-to-Sum and Sum-to-Product Formulas**

We can express a product of sines and cosines as sums and differences:

**Th: Product-To-Sum Formulas**

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

**Th: Sum-to-Product Formulas**

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

Ex. Express each of the following as a sum or difference:

1. $\sin 8x \sin 3x$
2. $\cos 7x \cos x$

Ex. Express each sum or difference as a product

1. $\sin 9x + \sin 5x$
2. $\cos 4x - \cos 3x$

Ex. Prove the identity: $\frac{\cos 3x - \cos 5x}{\sin 3x + \sin 5} = \tan x$

**Homework: Pg. 623 – 625 # 5-11o, 18-29o, 33, 37, 68**

**5.5 Trigonometric Equations**

Solving Trig Equations is exactly the same as solving an equation with a variable, with one added step:

| Solve for $x$ | Solve for $x$ |
|---|---|
| $2x-1=0$ | $2\sin x-1=0$ |
| $2x=1$ | $2\sin x=1$ |
| $x=\frac{1}{2}$ | $\sin x=\frac{1}{2}$ |

The difference between the two is that on the right side, you have solved for sin x. You want to find x. You need to go another step.

- You are looking for the angle whose sine is ½
- Use QRS to help…

$\sin x=\frac{1}{2}$

Q:

R:

S: 🡐 You get S first…it is positive since ½ is not – ½

- Now that you know S, you ask, "What quadrants is sin positive?
  - Quadrants I and II
- Calculate R: The angle whose sin is ½? (Ignore + or – now)
  - 30º
  - You can use your calculator to measure
  - Then calculate the angle in each quadrant based on the reference angle:
    Quad I: 30
    Quad II: 180-30 = 150

So $x$ = 30 and 150

Examples: Solve

1. $3\sin x-2=5\sin x-1$

2. $\tan 3x=1\text{ , }0\le x\le 2\pi$

3. $\sin\frac{x}{2}=\frac{\sqrt{3}}{2}$

4. $\sin\frac{x}{3}=\frac{1}{2}$

The equation may involve a quadratic or more than one trig function. If more than one trig function is involved, use the appropriate identity or formula to substitute:

Ex. Solve each of the following for $0 \leq x \leq 2\pi$

1. $4\sin^2 x - 1 = 0$

2. $\cos^2 x + 3\cos x - 1 = 0$

3. $3\sin x = 2\cos^2 x$

4. $\cos 2x + \cos x = 0$

5. $\tan x \sin^2 x = 3\tan x$

6. $2\cos 2x \sin 2x + \cos 2x = 0$

7. $2\cos^2 x + 3\sin x = 0$

8. $\cos 2x + 3\sin x - 2 = 0$

9. $\sin x \cos x = \frac{1}{2}$

10. $\sin x - \cos x = 1$

*If the angle is not a known value, use the inverse trig functions on your calculator to get the angle.*

**Homework: Pg. 636-637 #12-24(3's), 25-75(5's), 91, 95, 99, 106, 107, 111**