Chapter 8 – Matrices and Determinants

8.1 Matrix Solutions to Linear Systems

Def: A matrix (plural: *matrices*) is an arrangement of numbers in rows and columns (like a table).

Ex. $\begin{bmatrix} 23 & 12 & 11 & 18 & 3 & 11 \\ 9 & 2 & 10 & 5 & 23 & 16 \end{bmatrix}$

Each number in the matrix is called an element.

- Matrices are used to display information and to solve systems of linear equations

Def: An augmented matrix is a matrix that has a vertical bar separating the columns of the matrix into two groups.

- It is used as a shorthand way of writing a system of equations.

Ex.

System of Linear Equations	Augmented Matrix
$3x + y + 2z = 31$ $x + y + 2z = 19$ $x + 3y + 2z = 25$	$\left[\begin{array}{ccc\|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$
$x + 2y - 5z = -19$ $y + 3z = 9$ $z = 4$	$\left[\begin{array}{ccc\|c} 1 & 2 & -5 & -19 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 4 \end{array}\right]$

The circled elements are called the **main diagonal**.

To solve a system of linear equations using an augmented matrix, you want to change the left side of the augmented matrix to 1's on the main diagonal and 0's for the rest. The right side will be the solution.

Def: Matrix Row Operations – a method of converting numbers in rows of a matrix

- Basically the same as the elimination method, without the variables.

Operations that can be performed on an augmented matrix are:

1. Two rows can be interchanged. (Notation: $R_1 \leftrightarrow R_2$ - interchange Row 1 and Row 2)
2. The elements in any row can be multiplied/divided by any non-zero number. (Notation: $3R_2$)
3. Elements in any row may be added to or subtracted from corresponding elements in other rows.
   - These elements can be multiplied/divided by a constant before being added/subtracted.
   - Notation: $+2R_2$

Ex. Given the following matrix, perform the indicated row operations:

$$\left[\begin{array}{ccc|c} 3 & 18 & -12 & 21 \\ 1 & 2 & -3 & 5 \\ -2 & -3 & 4 & -6 \end{array}\right]$$

(a) $R_1 \leftrightarrow R_2$

(b) $\frac{1}{3}R_1$

(c) $R_3 = R_3 + 2R_2$

Gauss-Jordan Elimination Method: Solving a 3x3 System

1. Write the system as an augmented matrix
2. Use row operations to simplify the left side of the augmented matrix to the identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   - Start with getting the top left element to become 1. Then use row operations on row 1 to make the remaining elements in column 1 zero.
   - Using row operations (excluding using row 1), get the center element (row 2, column 2) to become 1. Use row operations on row 2 to make the remaining elements in column 2 zero
   - Repeat for the bottom right element.
3. Once the left side of the augmented matrix is the identity matrix, the right side will be the solution.

Ex. Use the Gauss-Jordan Elimination method to solve

$$3x + y + 2z = 31$$
$$x + y + 2z = 19$$
$$x + 3y + 2z = 25$$

Homework: Pg. 814-815 #1-8, 13, 15, 21, 25 , 35 (only use Gauss-Jordan)

8.2 Inconsistent and Dependent Systems and Their Applications

Ex. Solve using the Gauss-Jordan method

$$x - y - 2z = 2$$
$$2x - 3y + 6z = 5$$
$$3x - 4y + 4z = 12$$

Since the one of the rows of the matrix is all 0's (and the right side of the augmented matrix is a non-zero number), then the system has no solution. This system is considered an <u>inconsistent system</u>.

Ex. Solve using the Gauss-Jordan method

$$3x - 4y + 4z = 7$$
$$x - y - 2z = 2$$
$$2x - 3y + 6z = 5$$

Since one of the rows of the matrix is all 0's (including the right side), then it has an infinite number of solutions. This line results in an equation $0x + 0y + 0z = 0$, which is dependent on the other two equations. You can now drop it from the system, this gives you:

Rewriting the matrix as a system, we get:

Solving this system for *x* and *y*:

What if the system has more variables than equations?

Ex. Use Guass-Jordan Method to solve the following system

$$3x + 7y + 6z = 26$$
$$x + 2y + z = 8$$

Homework: Pg. 824 #3-24 (3's)

8.3 Matrix Operations and Their Applications – Day 1

Matrix Notation:

- Matrices are denoted with:
  - Capital letters: Such as A, B, C, …
  - A lower case letter enclosed in brackets

    Ex. $A = [a_{ij}]$
    - a_{ij} is called the general element of matrix A.
      - This refers to the element in the i[th] row and j[th] column.

        Ex. a_{32} is the element in matrix A in row 3, column 2.
- A matrix of order *m* x *n* has m rows and n columns.
  - If m = n, then the matrix has the same number or rows and columns. This is called a square matrix.

Ex. Consider the matrix $A = \begin{bmatrix} 3 & 2 & 0 \\ -4 & -5 & -\frac{1}{2} \end{bmatrix}$

(a) What is the order of matrix *A*?

(b) What is a_{23} and a_{12}?

Def: Two matrices are equal if and only if they have the same order and each corresponding elements are equal.

Mathematically: Matrix A = Matrix B if and only if (IFF) $a_{ij} = b_{ij}$ for i = 1, 2, …, m and j = 1, 2, … , n.

Ex. If $A = \begin{bmatrix} x & y+2 \\ z & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix}$ and A = B, find x, y, and z.

Matrix Addition and Subtraction

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of order $m \times n$

1.) $A + B = [a_{ij} + b_{ij}]$

2.) $A - B = [a_{ij} - b_{ij}]$

- Add/subtract corresponding elements of two matrices

Ex. Perform the indicated operation

1. $\begin{bmatrix} 0 & 5 & 3 \\ -2 & 6 & 8 \end{bmatrix} + \begin{bmatrix} -2 & 3 & 5 \\ 7 & -9 & 6 \end{bmatrix}$

2. $\begin{bmatrix} -6 & 7 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} -5 & 6 \\ 0 & -4 \end{bmatrix}$

Def: A matrix with all elements equal to 0 is called the zero matrix. Notation: [0]

Property: [A] + [0] = [A]

- The zero matrix [0] is called the additive identity for *m* x *n* matrices.

Def: If [A] + [B] = [0], then [B] is the additive inverse of [A].

Notation: $[-A]$ is the additive identity of [A].

Properties of Matrix Addition: If A, B, and C are *m* x *n* matrices and 0 is the *m* x *n* zero matrix, then

1. [A] + [B] = [B] + [A] – the Commutative Property
2. [A + B] + [C] = [A] + [B + C] – the Associative Property
3. [A] + [0] = [0] + [A] = A – Additive Identity Property
4. [A] + [-A] = 0 – Additive Inverse Property

Def: If $A = [a_{ij}]$ is a matrix of order *m* x *n* and *c* is a scalar (a number), then the matrix *c A* is the matrix given by

$$cA = [ca_{ij}]$$

- This is called scalar multiplication.

Ex. If $A = \begin{bmatrix} 2 & 5 \\ -4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ -1 & -4 \end{bmatrix}$, find

1. $-5B$
2. $2A + 3B$
3. $-A - 3B$

Properties of Scalar Multiplication: If A and B are *m* x *n* matrices, and *c* and *d* are scalars, then

1. $(cd)[A] = c(d[A])$
2. $1[A] = [A]$
3. $c([A] + [B]) = c[A] + c[B]$
4. $(c + d)[A] = c[A] + d[A]$

Ex. Solve for X in the matrix equation $2X + A = B$, when

$A = \begin{bmatrix} 1 & -5 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 5 \\ 9 & 1 \end{bmatrix}$

Homework: Pg. 838-839 #1-4, 7, 9-25o

8.3 Matrix Operations and Their Applications – Day 2

Matrix multiplication is not the same as matrix addition and subtraction.

- In order to multiply two matrices, A and B, the number of columns in matrix A = number of rows of matrix B
- To find the product of two matrices: multiply each element in the i^{th} row of A by the corresponding element in the j^{th} column in matrix B and then add the products.
- If [A] is an m x n matrix and [B] is an n x p matrix, then [A][B] is an m x p matrix.

Ex . Find [A][B] if

$$A = \begin{bmatrix} \boxed{1 \quad 2 \quad 4 \quad 0} \\ 6 & 7 & 8 & -1 \\ -2 & 4 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 6 & -1 \\ 0 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Each element of the answer is created by multiply corresponding elements in i^{th} row of A by the corresponding element in the j^{th} column in matrix B and then add the products

So the resulting matrix in the example will be a 3x3 matrix (a 3x4 times a 4x3 results in a 3x3).

So element in row 1, column 2 of answer will be the result of multiplying row 1 elements in A with elements in column 2 of B, then adding:

$$1(0) + 2(6) + 4(-2) + 0(1) = 4$$

Ex. If

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 \\ -1 & 4 \\ -2 & 3 \end{bmatrix}$$

Find (1) [A][B] (2) [B][A] (3) Does [A][B] = [B][A]?

Ex. Find the product (if possible) of

1. $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 3 & -1 & 6 \\ 0 & 5 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

<u>Properties of Matrix Multiplication</u>: If A, B, and C are *m* x *n* matrices and *c* is a scalar, then (if possible)

1. $([A][B])[C] = [A]([B][C])$
2. $[A]([B] + [C]) = [A][B] + [A][C]$
   $([A] + [B])[C] = [A][C] + [B][C]$
3. $c([A][B]) = (c[A])[B]$

Homework: Pg. 839 #18-26e, 27-43odd

Objectives:
1. Students will be able to prove matrices are inverses to each other.
2. Students will be able to find the inverse of a 2x2 matrix using two methods.

8.4 Multiplicative Inverses of Matrices (Day 1)

Def: The **multiplicative identity matrix** is a square matrix in which the main diagonal contains all 1's and the remainder of the matrix are 0's.

Ex. 2x2 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 3x3 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 4x4 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Notation: I_n where n is the size of the matrix.

Property: If A is an n x n matrix and I is the n x n identity matrix, then [A][I] = [I][A] = [A]

Ex. $\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ =

We use the multiplicative identity matrix to define and find the multiplicative inverse of [A].

Def: The multiplicative inverse of n x n matrix A, denoted $[A]^{-1}$, is n x n matrix such that:

$[A][A]^{-1} = I_n$ or $[A]^{-1}[A] = I_n$

******* NOTE *********

Only square matrices have multiplicative inverses!!!

Ex. Show that $\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$

Ex. Show that $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

Ex. Find the multiplicative inverse of

(a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$

Ex. Find the multiplicative inverse of $\begin{bmatrix} -6 & 4 \\ -3 & 2 \end{bmatrix}$

Def: A square matrix that has no multiplicative inverse is called a singular matrix.
A matrix that has a multiplicative inverse is called invertible or nonsingular.

Ok…..time for a shortcut.

To find the inverse of a 2x2 matrix:

If [A]= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $[A]^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Note: The matrix is invertible iff $ad - bc \neq 0$. If $ad - bc = 0$, then the matrix has no multiplicative inverse.

Ex. Using the formula, find the inverse of

(a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$

Homework: Pg. 853 #1-7, 9-17odd

8.4 Multiplicative Inverses of Matrices (Day 2)

Objective:
1. Students will be able to find the inverse of an nxn matrix.

<u>To find the inverse of an invertible 3x3 Matrix:</u>

1. Create an augmented matrix $[A|I_n]$, where I_n is the multiplicative identity matrix of the same order as [A]
2. Perform row operations on $[A|I_n]$ to obtain a new matrix of the form $[I_n|B]$. This is the Gauss-Jordan method.
3. The matrix [B] is $[A]^{-1}$

Ex. Find the inverse of

1. $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Homework: Pg. 853 #19-27odd

8.4 Multiplicative Inverses of Matrices (Day 3)

Objectives:
1. Students will be able to solve a system using inverse matrices.
2. Students will be able to use inverses to help in "coding"

- Matrix multiplication can be used to represent a system of linear equations.

Linear System	Matrix Form of the System
$a_1x + b_1y + c_1z = d_1$ $a_2x + b_2y + c_2z = d_2$ $a_3x + b_3y + c_3z = d_3$	$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The matrix form is set up the following way:

$$\begin{bmatrix} \text{matrix of coefficients} \\ n\times n \end{bmatrix}\begin{bmatrix} \text{matrix of variables} \\ n \times 1 \end{bmatrix} = \begin{bmatrix} \text{matrix of system's constants} \\ n\times 1 \end{bmatrix}$$

$$[A] \qquad [X] \qquad = \qquad [B]$$

Which is abbreviated $AX = B$. A is called the **coefficient matrix**. X and B are called **column matrices**. X contains the variables and B contains the constants of the system.

Ex.

Linear System	Matrix Form of the System
$x - y + z = 2$ $-2y + z = 2$ $-2x - 3y \quad = \frac{1}{2}$	$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ \frac{1}{2} \end{bmatrix}$

- The matrix equation $AX = B$ can be solved using A^{-1}, provided it exists:

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$I_nX = A^{-1}B$$
$$X = A^{-1}B$$

Therefore: If $AX = B$, then $X = A^{-1}B$

- So, to solve a system using A^{-1}:

  If $AX = B$ has a unique solution, then $X = A^{-1}B$. To solve a linear system of equations, multiply A^{-1} and *B* to find *X*.

Ex. Solve the system:

$$x - y + z = 2$$
$$-2y + z = 2$$
$$-2x - 3y \quad = \frac{1}{2}$$

Ex. Solve the system:

$$\begin{aligned} x \qquad\quad + 2z &= 6 \\ -x + 2y + 3z &= -5 \\ x - y \qquad\quad &= 6 \end{aligned}$$

<u>Applications of Matrix Inverses to Coding</u>:

- A *<u>cryptogram</u>* is a message written so that no one other than the intended recipient can understand it.
  - To encode a message (put it into code):
    - you assign a number to each letter:
      A = 1, B = 2, C = 3, … Z = 26, and a space = 0.
      Ex. FISH would be 6, 9, 19, 8
    - the numerical equivalent of the message is converted into a square matrix (by columns)
      - if you do not have enough numbers to make a square matrix, put zeros in any remaining space in the last COLUMN.
    - Select any square invertible matrix, called the coding matrix, the same size as the square matrix made earlier. Multiply the coding matrix by the square matrix that expresses the message numerically. This is the *<u>coded matrix</u>*.
    - Using the numbers, by columns, write the encoded message from the coded matrix.

Ex. Encode the word FISH using the matrix $\begin{bmatrix} 4 & -5 \\ 5 & -6 \end{bmatrix}$

Ex. Now decode it using the matrix $\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

Homework: Pg. 853-855 #20-28e,29-35o, 38, 39, 41, 47, 48, 53, 54 (use a calculator)

8.5 Determinants and Cramer's Rule

Def: A determinant is an algebraic expression that transfers a square matrix into a scalar.

To find the determinant of a 2x2 matrix: (Second-order determinant)

The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$

The notation for the determinant is $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Ex. Find the determinants for each of the following

1. $\begin{bmatrix} 5 & 6 \\ 7 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 2 & 3 \\ -4 & -5 \end{bmatrix}$
3. $\begin{bmatrix} 10 & 9 \\ 6 & 5 \end{bmatrix}$
4. $\begin{bmatrix} 4 & 3 \\ -5 & -8 \end{bmatrix}$

Determinants can be used to help solve a 2x2 system of equations: (**Cramer's Rule**)

If

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Then

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

provided $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

ex. Use Cramer's Rule to solve the system

1. $5x - 4y = 2$
   $6x - 5y = 1$

2. $5x + 4y = 12$
   $3x + 6y = 24$

Determinant of a 3x3 Matrix: (Third-order determinant)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - id$$

It can also be found with mini-determinants:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

Ex. Find the value of the determinant:

1. $\begin{vmatrix} 4 & 1 & 0 \\ -9 & 3 & 4 \\ -3 & 8 & 1 \end{vmatrix}$

2. $\begin{vmatrix} 2 & 1 & 7 \\ -5 & 6 & 0 \\ -4 & 3 & 1 \end{vmatrix}$

Homework: Pg. 866 #1-31 odd

8.5 Determinants and Cramer's Rule (Day 2)

Ex. Find the value of the determinant:

1. $\begin{vmatrix} 9 & 5 & 0 \\ -2 & -3 & 0 \\ 1 & 4 & 2 \end{vmatrix}$

2. $\begin{vmatrix} 6 & 4 & 0 \\ -3 & -5 & 3 \\ 1 & 2 & 0 \end{vmatrix}$

Recall: **Cramer's Rule**:

If

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Then

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

provided $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

Notation: For ease of writing, we can write in an abbreviated notation $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$ where

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \qquad D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

To get D_x, simply replace the x-coefficients (the a's) with the constants(the c's). D_y is found in a similar fashion replacing the *y*-coefficients (the b's) with the constants (the c's). D is simply the coefficient determinant.

This general idea can be adapted for 3x3 System:

Cramer's Rule for a 3x3 System of Linear Equations:

If

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then

$$x = \frac{D_x}{D}, \; y = \frac{D_y}{D}, \text{ and } z = \frac{D_z}{D} \qquad \text{where } D \neq 0$$

The four determinants needed are:

$$D = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, D_x = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}, D_y = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}, \text{and } D_z = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$$

Ex. Solve the system using Cramer's Rule:

1. $x + 2y - z = -4$
   $x + 4y - 2z = -6$
   $2x + 3y + z = 3$

2. $3x - 2y + z = 16$
   $2x + 3y - z = -9$
   $x + 4y + 3z = 2$

Note: If $D = 0$ and any of the numerator determinants are not 0, then the system in inconsistent and has NO SOL^{n}

If $D = 0$ and all of the numerator determinants are 0, then the system is dependent → infinite # of solutions

Homework: Pg. 866-867 #28-32e, 33, 37, 39, 45, 49, 51, 53-58